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Note

A Short Proof of the Two-Commodity Flow Theorem

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We give a new proof of the following theorem, due to T. C. Hu [3].

THEOREM. *Let e_1, e_2 be distinct edges of a directed graph $G = (V, E)$, and let c: $E \to \mathbb{R}^+$ be given. Then the following are equivalent:* —

(i) *there are circulations F_1, F_2 of G such that*

$$|F_1(e_1)| = c(e_1),\ F_2(e_1) = 0$$
$$F_1(e_2) = 0,\ |F_2(e_2)| = c(e_2)$$
$$|F_1(e)| + |F_2(e)| \leqslant c(e) \qquad \text{for} \quad e \in E - \{e_1, e_2\}$$

(ii) *for every $X \subseteq V$,*

$$c(\partial(X) \cap \{e_1, e_2\}) \leqslant c(\partial(X) - \{e_1, e_2\}).$$

[Here $\mathbb{R}$ and $\mathbb{R}^+$ are the sets of real and non-negative real numbers respectively. A *circulation* F of G is a map F: $E \to \mathbb{R}$ such that for each vertex $v \in V$,

$$\Sigma(F(e)\colon e \in \partial^-(v)) = \Sigma(F(e)\colon e \in \partial^+(v))$$

where $\partial^-(v)$ and $\partial^+(v)$ are the sets of edges directed respectively into and out of v. For $X \subseteq V$, $\partial(X)$ denotes the set of edges of G with precisely one end in X. For $E' \subseteq E$, $c(E')$ denotes $\Sigma_{e \in E'}\, c(e)$.]

Proof. That (i) implies (ii) is obvious; we prove the converse. By Hoffman's circulation theorem [1, 2] (or by easy manipulation of the max-flow min-cut theorem itself [1]) there is a circulation H_1 of G such that

$$H_1(e_1) = c(e_1)$$
$$H_1(e_2) = c(e_2)$$
$$-c(e) \leqslant H_1(e) \leqslant c(e)\ (e \in E - \{e_1, e_2\}).$$

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Similarly, there is a circulation H_2 such that

$$H_2(e_1) = c(e_1)$$
$$H_2(e_2) = -c(e_2)$$
$$-c(e) \leqslant H_2(e) \leqslant c(e) \; (e \in E - \{e_1, e_2\}).$$

Put $F_1 = \frac{1}{2}(H_1 + H_2)$, $F_2 = \frac{1}{2}(H_1 - H_2)$; we claim that these satisfy (i). Certainly

$$| F_1(e_1)| = c(e_1), F_2(e_1) = 0$$
$$F_1(e_2) = 0, | F_2(e_2)| = c(e_2);$$

and for $e \in E - \{e_1, e_2\}$,

$$\begin{aligned} | F_1(e)| + | F_2(e)| &= | \tfrac{1}{2}(H_1(e) + H_2(e))| + | \tfrac{1}{2}(H_1(e) - H_2(e))| \\ &= \max (| H_1(e)|, | H_2(e)|) \\ &\leqslant c(e) \end{aligned}$$

as required.

References

1. L. R. Ford, Jr. and D. R. Fulkerson, "Flows in Networks," (Princeton Univ. Press, Princeton, N.J., 1962).
2. A. J. Hoffman, Some recent applications of the theory of linear inequalities to extremal combinatorial analysis, *in* "Proc. Symposia on Applied Math." Vol. 10, 1960.
3. T. C. Hu, "Multi-commodity network flows," *Operations Res.* **11** (1963), 344–360.